HEATING OF A MASSIVE SPHERICAL PARTICLE BY RADIATION AND CONVECTION

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An approximate analytical solution of the nonlinear problem of heating of a spherical body by radiation and convection is obtained.

The problem of calculating the unsteady temperature field in spherical particles heated by radiation and convection arises in the investigation and adjustment of compartment and rotary furnaces, and certain other heating devices [1]. A similar problem also occurs in mass-transfer theory when the period of initial heating of particles of finite size is being estimated [2].

The actual problem in the case of spherical symmetry reduces to solution of the Fourier equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial T}{\partial r} = \frac{1}{a}\cdot\frac{\partial T}{\partial \tau} \tag{1}$$

with the following boundary conditions:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0; \quad \lambda\left.\frac{\partial T}{\partial r}\right|_{r=R} = [\alpha(T_p - T) + c_s(T_s^4 - T^4)]|_{r=R}, \tag{2}$$

$$T|_{\tau=0} = T_0.$$

We will assume henceforth that the values of T_p and T_s are independent of time, and T_0 is independent of the coordinate. Introducing the dimensionless numbers and simplices

$$\mathrm{Fo} = \frac{a\tau}{R^2}, \quad \mathrm{Bi} = \frac{\alpha R}{\lambda}, \quad I = \frac{c_s T_s^3 R}{\lambda},$$
$$\varphi = \frac{r}{R}, \quad \theta = \frac{T}{T_s}, \quad \theta_0 = \frac{T_0}{T_s}, \quad \theta_p = \frac{T_p}{T_s}, \tag{3}$$

we write the initial equations in dimensionless form,

$$\frac{\partial^2\theta}{\partial\varphi^2} + \frac{2}{\varphi}\cdot\frac{\partial\theta}{\partial\varphi} = \frac{\partial\theta}{\partial \mathrm{Fo}},$$
$$\left.\frac{\partial\theta}{\partial\varphi}\right|_{\varphi=0} = 0; \quad \left.\frac{\partial\theta}{\partial\varphi}\right|_{\varphi=1} = [\mathrm{Bi}(\theta_p - \theta) + J(1-\theta^4)]_{\varphi=1}, \tag{4}$$
$$\theta|_{Fo=0} = \theta_0.$$

To generalize the problem to some extent, we will proceed from a more general nonlinear boundary condition on the surface of the sphere,

$$\left.\frac{\partial\theta}{\partial\varphi}\right|_{\varphi=1} = q(\theta)|_{\varphi=1}, \tag{5}$$

All-Union Scientific-Research and Planning Institute for Machining of Minerals, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 5, pp. 926-932, November, 1975. Original article submitted March 12, 1974.

where q( ) is an analytical function of its argument and, hence, can be put in the form of a power series.

Using the Laplace operator method and the first and third boundary conditions of (4), we obtain the following expression for the image $\theta(\varphi, p)$ of the temperature:

$$\Theta(\varphi, p) = \left[\Theta(1, p) - \frac{\theta_0}{p}\right] \frac{\operatorname{sh}\sqrt{p}\,\varphi}{\varphi \operatorname{sh}\sqrt{p}} + \frac{\theta_0}{p}.$$

The aim of the following investigation was to obtain approximate solutions of the considered nonlinear problem, valid either at low, or high Fo.

Solutions of Problem for Small Fo

We differentiate expression (6) with respect to φ and then put $\varphi = 1$. Then in view of boundary condition (5), we can write

$$Q = [\Theta(1, p)p - \theta_0]\left(\frac{\operatorname{cth}\sqrt{p}}{\sqrt{p}} - \frac{1}{p}\right), \tag{7}$$

where Q(p) is the image of $q(\theta)|_{\varphi=1}$. Converting now to the original, we obtain

$$q(\theta_0 + u) = \frac{1}{\sqrt{\pi}} \int\limits_0^{\mathrm{Fo}} \frac{du(\eta)}{d\eta} \vartheta(\mathrm{Fo} - \eta)d\eta - u. \tag{8}$$

Here $u = \theta|_{\varphi=1} - \theta_0$ is the excess temperature on the sphere surface, and $\vartheta(\mathrm{Fo})$ is a theta function [3], which for small Fo can be written in the form

$$\vartheta = \frac{1}{\sqrt{\mathrm{Fo}}}\left\{1 + 2\sum_{k=1}^{\infty} \exp\left(-\frac{k^2}{\mathrm{Fo}}\right)\right\}. \tag{9}$$

Relation (8) is an exact integrodifferential equation for determination of the excess temperature u(Fo). Its effective approximate solution, however, can be obtained only for small Fo. For this purpose we neglect the terms in (9) which decrease exponentially when $\mathrm{Fo} \to 0$. We can then replace (8) by the equation

$$q(\theta_0 + u) + u = \frac{1}{\sqrt{\pi}} \int\limits_0^1 [\mathrm{Fo}(u) - \mathrm{Fo}(u\xi)]^{-1/2} d\xi \tag{10}$$

for the unknown Fo(u). The new variable of integration ξ is

$$\xi = \frac{u(\eta)}{u(\mathrm{Fo})}. \tag{11}$$

The integral equation (10) allows the following approximate solution, which is suitable for small u:

$$\mathrm{Fo}(u) = \frac{0.786}{q_0^2}u^2 - \frac{1.24(q_1+1)}{q_0^3}u^3 - \left[1.05\frac{q_2}{q_0^3} - 3.46\frac{(q_1+1)}{q_0^4}\right]u^4 - \\ - \left[0.93\frac{q_3}{q_0^3} - 2.85\frac{(q_1+1)q_2}{q_0^4} + 7.04\frac{(q_1+1)^3}{q_0^5}\right]u^5 + u^6 \ldots, \tag{12}$$

where the numbers q_0, q_1, q_2, ... are the coefficients of the expansion of function $q(\theta)$ in a Taylor series near the point $\theta = \theta_0$, i.e.,

$$q_i = \frac{1}{i!}\cdot\frac{d^iq}{d\theta^i}\bigg|_{\theta=\theta_0}, \quad i = 0, 1, 2, \ldots \tag{13}$$

The correctness of solution (12) can be verified by direct substitution in Eq. (10). For purely convective heat transfer (J = 0), we obtain instead of (12)

$$\mathrm{Fo}(u) = \frac{0.786}{\mathrm{Bi}^2}u^2 - \frac{1.24(1-\mathrm{Bi})}{\mathrm{Bi}^3}u^3 + 3.46\frac{(1-\mathrm{Bi})^2}{\mathrm{Bi}^4}u^4 - 7.04\frac{(1-\mathrm{Bi})^3}{\mathrm{Bi}^5}u^5 + u^6 \ldots \quad (\theta_p = 1). \tag{14}$$

A direct comparison of solution (14) with the known results of the exact solution for small Fo [4] shows its high accuracy.

To evaluate the region of applicability of solution (12) we determine the order of the discrepancy Δ after its substitution in the exact equation (8). If for simplicity we keep only the first, maximum term in series (9), we obtain as a result

$$\Delta=\frac{2}{\sqrt{\pi}}\int_0^1[\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)]^{-1/2}\exp\{-[\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)]^{-1}\}d\xi<\frac{2}{\sqrt{\pi}}\exp[-\mathrm{Fo}(u)^{-1}]\int_0^1[\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)]^{-1/2}d\xi. \quad (15)$$

Hence, taking Eq. (10) into account, we easily obtain from the condition $\Delta \ll 1$ the following sufficient inequality for applicability of solution (12):

$$\mathrm{Fo}\ll\frac{1}{\ln 2(q+u)}. \quad (16)$$

The temperature field θ(, Fo) for small Fo($p \to \infty$) can conveniently be determined from the following approximate analog of (6):

$$\Theta=\frac{1}{\varphi}\left[\Theta(1,p)-\frac{\theta_0}{p}\right]\{\exp[-\sqrt{p}(1-\varphi)]-\exp[-\sqrt{p}(1+\varphi)]\}+\frac{\theta_0}{p}. \quad (17)$$

Conversion from this relation to the original gives

$$\theta=\theta_0+\frac{u}{\varphi}\int_0^1\left[\mathrm{erf}\frac{1+\varphi}{2\sqrt{\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)}}-\mathrm{erf}\frac{1-\varphi}{2\sqrt{\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)}}\right]d\xi. \quad (18)$$

The temperature at the center of the sphere (φ = 0) is obtained from (18) after evaluation of an indeterminate form of the 0/0 type and taking into account that here $\mathrm{Fo} \ll 1$,

$$\theta_c=\theta_0+\frac{2u}{\sqrt{\pi}}\int_0^1[\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)]^{-1/2}\exp\left\{-\frac{1}{4}[\mathrm{Fo}(u)-\mathrm{Fo}(u\xi)]^{-1}\right\}d\xi. \quad (19)$$

It is characteristic that this value has an error of the order of that obtained in calculation of u [see (15)]. Finally, to calculate the average temperature $\bar{\theta}$(Fo) over the volume we can use the exact balance relation

$$\frac{d\bar{\theta}}{d\mathrm{Fo}}=3q(u). \quad (20)$$

Solution of Problem for Large Fo

We put the heat flux on the sphere surface in the form

$$q=q^{(1)}v+q_*, \quad (21)$$

where $q^{(1)}, q^{(2)}, \ldots,$ is the dimensionless deviation of the varying temperature on the sphere surface from its steady value, determined from the condition: $q(\theta_*) = 0$,

$$q_*=q^{(2)}v^2+q^{(3)}v^3+v^4,\ldots, \quad (22)$$

and the numbers $q^{(1)}$, $q^{(2)}$, ..., determined from the formula

$$q^{(i)}=(-1)^{(i)}\frac{1}{i!}\cdot\frac{d^iq}{d\theta^i}\bigg|_{\theta=\theta_*}\quad(i=1,2,\ldots,) \quad (23)$$

are the coefficients of the expansion of q in a power series near the point $\theta = \theta_*$. We now express the Laplace image of the flux in accordance with (21) and (7) and from the obtained relation we determine the image V for v. If the obtained expression is substituted in (6), we finally obtain the following formula:

$$\Theta=\frac{\theta_0}{p}+\frac{\mathrm{sh}\sqrt{p}\varphi}{\sqrt{p}\varphi}\cdot\frac{\frac{\theta_*-\theta_0}{p}q^{(1)}+Q_*}{\mathrm{ch}\sqrt{p}+(q^{(1)}-1)\frac{\mathrm{sh}\sqrt{p}}{\sqrt{p}}}, \quad (24)$$

where Q_* is the image of q_*. Converting directly from (24) to the original, we arrive at the following exact expression for the temperature field:

$$\theta = \theta_* - v_0(\mathrm{Fo}, \varphi) + \int_0^{\mathrm{Fo}} q_*[v(\eta)]\,\Phi(\mathrm{Fo} - \eta, \varphi)\,d\eta. \tag{25}$$

Here

$$\begin{gathered} v_0 = q^{(1)}(\theta_* - \theta_0)\sum_{n=1}^{\infty}\frac{A_n(\varphi)}{\gamma_n^2}\exp(-\gamma_n^2\mathrm{Fo}), \\ \Phi(\mathrm{Fo}, \varphi) = \sum_{n=1}^{\infty} A_n(\varphi)\exp(-\gamma_n^2\mathrm{Fo}), \\ A_n(\varphi) = \frac{2\gamma_n}{\varphi}\cdot\frac{\sin\gamma_n\cdot\sin\gamma_n\varphi}{\gamma_n - \sin\gamma_n\cdot\cos\gamma_n}, \end{gathered} \tag{26}$$

and the numbers $\gamma_1, \gamma_2, \ldots$ are successively increasing positive roots of the transcendental equation

$$\mathrm{tg}\gamma_n = \frac{\gamma_n}{1 - q^{(1)}}. \tag{27}$$

It is easy to verify that in the absence of radiation ($J = 0$, $q_* = 0$, $q^{(1)} = \mathrm{Bi}$, $\theta_p = 1$) solution (25) is the same as the known solution [4]. In the considered nonlinear problem the use of (25) leads to the following nonlinear integral equation for the unknown $v(\mathrm{Fo})$:

$$v = v_0(\mathrm{Fo}) - \int_0^{\mathrm{Fo}} [q^{(2)}v^2(\eta) + q^{(3)}v^3(\eta) + v^4, \ldots,]\,\Phi(\mathrm{Fo} - \eta)\,d\eta. \tag{28}$$

Equation (28) is obtained directly by putting $\varphi = 1$ in (25). The sense of the coefficients contained in it is obvious; we note that when $n \to \infty$, $\gamma_n \to \pi/2 + n\pi$ and $A_n(1) \to 2$. Hence, function $\Phi(\mathrm{Fo})$ has a ϑ-type form and when $\mathrm{Fo} \to 0$ behaves like $1/\sqrt{\mathrm{Fo}}$. Accordingly, at the upper limit $\mathrm{Fo} = \eta$ the quadrature (28) has a singularity similar to that present at small Fo.

Equation (28) is very suitable for determination of the temperature at the surface of the body at large Fo. In fact, if $\mathrm{Fo} \to \infty$, $v \to 0$. Hence, the quadrature in this equation has a higher order of smallness and in the zero approximation we naturally put $v = v_0$. Higher approximations $v_1, v_2, \ldots,$ to v can be sought with the aid of the usual iterations

$$v_{s+1} = v_0 - \int_0^{\mathrm{Fo}} q_*[v_s(\eta)]\,\Phi(\mathrm{Fo} - \eta)\,d\eta \quad (s = 0, 1, \ldots,). \tag{29}$$

In a similar way we determine successive approximations to the temperature at any point of the sphere in accordance with (25) from the formulas

$$\theta_s = \theta_* - v_0(\mathrm{Fo}, \varphi) + \int_0^{\mathrm{Fo}} q_*[v_s(\eta)]\Phi[(\mathrm{Fo} - \eta), \varphi]\,d\eta. \tag{30}$$

In particular, the first approximation θ_1, according to (30), is

$$\theta_1 = \theta_* - v_0(\mathrm{Fo}, \varphi) + q^{(1)^2}(\theta_* - \theta_0)\sum_{i, j, n=1}^{\infty}\frac{A_i(1)A_j(1)A_n(\varphi)}{\gamma_i^2\gamma_j^2}\times$$
$$\times\left\{q^{(2)}\frac{\exp[-\gamma_n^2\mathrm{Fo}] - \exp[-(\gamma_i^2 + \gamma_j^2)\mathrm{Fo}]}{\gamma_i^2 + \gamma_j^2 - \gamma_n^2}\right\} +$$

$$+q^{(1)}q^{(3)}(\theta_*-\theta_0)\sum_{l=1}^{\infty}\frac{A_l(1)}{\gamma_l^2}\;\frac{\exp[-\gamma_n^2\mathrm{Fo}]-\exp[-(\gamma_i^2+\gamma_j^2+\gamma_l^2)\mathrm{Fo}]}{\gamma_i^2+\gamma_j^2+\gamma_l^2-\gamma_n^2}. \tag{31}$$

We note that expression (31) for determination of the first approximation $\theta_1(\varphi, \mathrm{Fo})$ contains terms with small denominators, i.e., those for which the quantities $\gamma_i^2 + \gamma_j^2 - \gamma_n^2$, $\gamma_i^2 + \gamma_j^2 + \gamma_l^2 - \gamma_n^2$, ... are as small as desired or even, at particular values of $q^{(1)}$, equal to zero. An investigation of the indeterminate form of the 0/0 type arising in this case shows that these terms behave approximately, or exactly, like Fo exp $(-\gamma_n^2\mathrm{Fo})$. However, since the number n for such terms with secular members is at least two, we can state that for sufficiently large Fo the difference between the initial and first approximations is insignificant.

Since the value of γ_1^2 is always much less than γ_2^2, γ_3^2, ..., for sufficiently large Fo we can replace (31) by the following regular formula:

$$\theta_1=\theta_*-F(\varphi)\exp(-\gamma_1^2\mathrm{Fo}). \tag{32}$$

Here we have introduced the symbol

$$F(\varphi)=q^{(1)}(\theta_*-\theta_0)\left\{\frac{1}{\gamma_1^2}-q^{(1)}(\theta_*-\theta_0)\sum_{i,\,j=1}^{\infty}\frac{A_i(1)A_j(1)}{\gamma_i^2\gamma_j^2}\times\right.$$

$$\left.\times\left[\frac{q^{(2)}}{\gamma_i^2+\gamma_j^2-\gamma_1^2}+\sum_{l=1}^{\infty}\frac{A_l(1)}{\gamma_l^2}\;\frac{q^{(1)}q^{(3)}(\theta_*-\theta_0)}{\gamma_i^2+\gamma_j^2+\gamma_l^2-\gamma_1^2}+\dots\right]\right\}A_1(\varphi). \tag{33}$$

It is easy to verify that in the linear problem (in the absence of radiation) the quadrature in expression (25) disappears, and the expression for the first approximation (32) represents the course of heat transfer in the regular regime of the first kind [4]. It should be noted that in the general nonlinear case (heating by radiation and convection) the change in temperature in the first approximation (32) has also a constant logarithmic derivation and a similar cross-sectional distribution, but with a large gradient, which depends on the nonlinear terms.

The approximate methods developed in this paper for the obtention of solutions for low and high Fo can easily be extended to the cases of a plate and cylinder.

NOTATION

T, temperature, °K; T_0, initial temperature of body; T_p, temperature of flux; T_s, emitter temperature; r, variable radius; R, radius of sphere; τ, time; a, thermal diffusivity; λ, thermal conductivity; α, convective heat-transfer coefficient; c_s, reduced radiation coefficient.

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